MOTION OF A CRACK AT CONSTANT VELOCITY

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We solve the plane self-similar Broberg problem [1] of the propagation of a crack from zero at constant velocity under the influence of a constant tensile stress p. As in [2], we make two assumptions relative to the end region of the crack surface (Fig. 1) and the distribution of cohesion forces:

1) the end region of the crack surface, where the existence of cohesion forces is important, expands at a constant velocity $(1-\alpha)$ V, where $0 < \alpha < 1$;

2) the cohesion force distribution function for the end region of the crack is approximated by the constant q_0.

We have the equations of motion

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial\tau^2}, \quad \Delta\psi = \frac{1}{k^2}\frac{\partial^2\psi}{\partial\tau^2}, \quad \tau = c_1 t, \quad k^2 = \frac{c_2^2}{c_1^2} = \frac{1-2\nu}{2-2\nu} < 1$$

Here, c_1 is the longitudinal wave velocity and c_2 is the transverse wave velocity.

The functions φ and ψ are given by the relations

$$\varphi(x, z, \tau) = \operatorname{div} \mathbf{u}(x, z, \tau), \quad \psi(x, z, \tau)\,\mathbf{y} = \operatorname{rot} \mathbf{u}(x, z, \tau)$$

where $\mathbf{u}(x, z, \tau)$ is the displacement vector. It is required to find the solution of the problem of the theory of elasticity with zero initial data and the boundary conditions

$$\sigma_{zz} = \begin{cases} -p & |x| < \gamma_0\tau \\ -(p-q_0) & \gamma_0\tau < |x| < \beta_0\tau \end{cases} \quad \text{at} \quad z=0, \ \tau > 0$$
$$\sigma_{zx} = 0 \quad \text{at} \quad |x| < \infty \ z = 0, \ \tau > 0 \quad (\beta_0 = V_0 / c_1, \ \gamma_0 = \alpha\beta_0)$$
$$u_z = 0 \quad \text{at} \quad |x| > \beta_0\tau$$

It is required that the tensile stress σ_{zz} be finite in the neighborhood of the tip of the moving crack.

In constructing the solution of the problem we follow Broberg [1]. We start by solving the two problems of finding the $\partial^2 u_z/\partial\tau^2$ of points of the surface z = 0 under the action of the loads

$$\sigma_{zz} = \begin{cases} -p, & |x| < \gamma_0\tau \\ -(p_0 - q_0), & \gamma_0\tau < |x| < \beta_0\tau \\ 0, & |x| > \beta_0\tau \end{cases} \quad \text{at} \quad z = 0, \ \tau > 0 \tag{1}$$
$$\sigma_{zx} = 0 \quad \text{at} \quad |x| < \infty, \ z = 0, \ \tau > 0$$

and

$$d\sigma_{zz} = \begin{cases} 0, & |x| < \beta_0\tau \\ -q'(\beta)\,d\beta, & \beta_0\tau < |x| < \beta\tau \\ 0, & |x| > \beta\tau \end{cases} \quad \text{at} \quad z = 0, \ \tau < 0 \tag{2}$$
$$\sigma_{zx} = 0 \quad \text{at} \quad |x| < \infty, \ z = 0, \ \tau > 0$$

The stresses given by (1) and (2) are superposed in such a way that $\partial^2 u_z/\partial\tau^2 = 0$ for $|x| > \beta_0\tau$ at z = 0, which leads to a singular integral equation for the function $q'(\beta)$; as distinct from the analogous equation in [1], we have

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$$
a(r)\varphi(r)+\frac{1}{\pi}\int\limits_{\beta_0^2}^{1}\frac{b(s)\varphi(s)}{s-r}\,ds=K(r) \tag{3}
$$

$$
x^2/\tau^2=r, \qquad \beta^2=s, \qquad q'(\beta)=\varphi(s), \qquad b(r)=r-\beta_0^2
$$

$$
a(r)=-\frac{\operatorname{Im}F(r^{-1})}{\operatorname{Re}F(r^{-1})}(r-\beta_0^2)=\begin{cases}0 & (\beta_0^2<r<k^2)\\ g(r)(r-\beta_0^2) & (k^2<r<1)\end{cases}
$$

$$
g(r)=\frac{4k^3\sqrt{(1-r)(r-k^2)}}{(r-2k^2)^2}
$$

$$
K(r)=\frac{2(p-q_0)}{\pi}\beta_0+\frac{1}{\pi}\int\limits_{\beta_0^2}^{1}\frac{\sqrt{s}-\beta_0}{\sqrt{s}}\varphi(s)\,ds+\frac{2q_0\gamma_0}{\pi}\,\frac{r-\beta_0^2}{r-\gamma_0^2}=K+\frac{2q_0\gamma_0}{\pi}\,\frac{r-\beta_0^2}{r-\gamma_0^2}
$$

$$
F(r)=\frac{\sqrt{r^2-1}}{(^1/_2\,k^{-2}-r^2)^2+r^2\sqrt{1-r^2}\sqrt{k^{-2}-r^2}}
$$

The general solution of Eq. (3) is written in the same form as in [1], except that the function K(r) everywhere replaces the constant K. As a result of integration and algebraic transformations we obtain

$$
\varphi(r)=Q\frac{f(r)}{r}+Q_1\frac{f(r)}{r-k_s^2}-A\frac{f(r)\,[r+(k_s^2-\gamma_0^2)]}{r(r-\gamma_0^2)} \qquad (\beta_0^2<r<k^2)
$$

$$
\varphi(r)=Q\frac{f_1(r)}{r}+Q_1\frac{f_1(r)}{r-k_s^2}-A\frac{f_1(r)\,[r+(k_s^2-\gamma_0^2)]}{r(r-\gamma_0^2)} \qquad (k^2<r<1)
$$

$$
f(r)=\frac{(r-2k^2)^2-4k^3\sqrt{(1-r)(k^2-r)}}{\sqrt{(1-r)(r-\beta_0^2)^3}}, \qquad k_s^2=\frac{c_s^2}{c_1^2}
$$

$$
f_1(r)=\frac{(r-2k^2)^2}{\sqrt{(1-r)(r-\beta_0^2)^3}}, \qquad A=\frac{2q_0\gamma_0}{\pi}\,\frac{(\gamma_0^2-\beta_0^2)\sqrt{(1-\gamma_0^2)(\beta_0^2-\gamma_0^2)}}{k_s^2(k_s^2-\gamma_0^2)}
$$

Here, c_s is the velocity of the Rayleigh surface waves.

Broberg [1] showed that for satisfaction of the condition $u_z = 0$ at $|x| > \beta_0\tau$ it is necessary that $Q_1 = 0$; this leads to the equation

$$
Q=\frac{2(p-q_0)}{\pi}\beta_0+\frac{1}{\pi}\int\limits_{\beta_0^2}^{1}\frac{\sqrt{s}-\beta_0}{\sqrt{s}}\varphi(s)\,ds+\frac{2q_0\gamma_0}{\pi}+A \tag{4}
$$

Assuming that constant Q is known, we find σ_{zz} for $|x| > \beta_0\tau$. Since

$$
d\sigma_{zz}=-q'(\beta)\,d\beta, \qquad q'(\beta)=\varphi(r)
$$

we have

$$
\sigma_{zz}=-\frac{1}{2}\int\limits_{x^2/\tau^2}^{1}\frac{\varphi(r)}{\sqrt{r}}\,dr
$$

Setting $|x|/\tau = \xi$, after integration with $\beta_0 < \xi < k$ we obtain

$$
\begin{aligned}
\sigma_{zz}(\xi)={}&\frac{Q}{\beta_0^2(1-\beta_0^2)}\Big\{\frac{\xi}{\beta_0^2\sqrt{\xi^2-\beta_0^2}}\,[4k^3(1-\beta_0^2)\sqrt{k^2-\xi^2}-(\beta_0^2-2k^2)^2\sqrt{1-\xi^2}]\\
&+\frac{4k^3\sqrt{\xi^2-\beta_0^2}}{\beta_0^2\xi}\,[\sqrt{k^2-\xi^2}-k\sqrt{1-\xi^2}]-[\beta_0^2(1-4k^2)+4k^4]\,F(\lambda_1,\ q_1)\\
&+\frac{1}{\beta_0^2}\,[\beta_0^4-4k^2(1+k^2)\beta_0^2+8k^4]\,E(\lambda_1,\ q_1)+4k^2(1-\beta_0^2)\,F(\lambda_k,\ q_k)\\
&-\frac{8k^4}{\beta_0^2}(1-\beta_0^2)\,E(\lambda_k,\ q_k)\Big\}+\frac{A}{(\beta_0^2-\gamma_0^2)(1-\beta_0^2)}\Big\{\frac{\xi}{\beta_0^2\sqrt{\xi^2-\beta_0^2}}\,[4k^3(1-\beta_0^2)\sqrt{k^2-\xi^2}\\
&-(\beta_0^2-2k^2)^2\sqrt{1-\xi^2}]-[\beta_0^2(1-4k^2)-\gamma_0^2(1-\beta_0^2)+4k^4]\,F(\lambda_1,\ q_1)\\
&+\frac{(\beta_0^2-2k^2)^2}{\beta_0^2}\,E(\lambda_1,\ q_1)+4k^2(1-\beta_0^2)\,F(\lambda_k,\ q_k)-\frac{4k^4(1-\beta_0^2)}{\beta_0^2}\,E(\lambda_k,\ q_k)\\
&-4k^2(1-\beta_0^2)\,\Pi(\lambda_k,\ -\varkappa_k,\ q_k)+\frac{(1-\beta_0^2)}{(1-\gamma_0^2)}(\gamma_0^2-2k^2)^2\,\Pi(\lambda_1,\ -\varkappa_1,\ q_1)\Big\}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&+\frac{A(k_s^2-\gamma_0^2)}{\beta_0^2(\beta_0^2-\gamma_0^2)(1-\beta_0^2)}\Big\{\frac{\xi}{\beta_0^2\sqrt{\xi^2-\beta_0^2}}\,[4k^3(1-\beta_0^2)\sqrt{k^2-\xi^2}-(\beta_0^2-2k^2)^2\sqrt{1-\xi^2}]\\
&+\frac{4k^3(\beta_0^2-\gamma_0^2)(1-\beta_0^2)}{\xi\gamma_0^2\beta_0^2}\sqrt{\xi^2-\beta_0^2}\,[k\sqrt{1-\xi^2}-\sqrt{k^2-\xi^2}]-(\beta_0^2-2k^2)^2\,F(\lambda_1,\ q_1)
\end{aligned}
$$

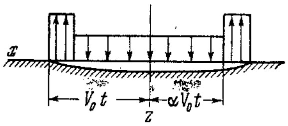

Fig. 1

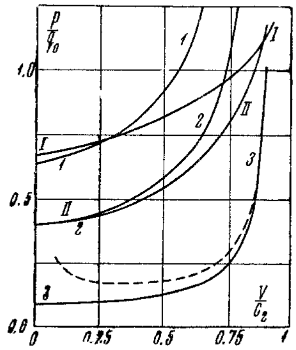

Fig. 2

$$+\frac{\beta_0^4(\gamma_0^2+4k^4)-4\beta_0^2k^2(\gamma_0^2+k^2+k^2\gamma_0^2)+8k^4\gamma_0^2}{\beta_0^2\gamma_0^2}E(\lambda_1,\ q_1)$$
$$+4k^2(1-\beta_0^2)F(\lambda_k,\ q_k)-\frac{4k^4(2\gamma_0^2-\beta_0^2)(1-\beta_0^2)}{\beta_0^2\gamma_0^2}E(\lambda_k,\ q_k) \tag{5}$$
$$+\frac{\beta_0^2}{\gamma_0^2}\frac{1-\beta_0^2}{1-\gamma_0^2}(\gamma_0^2-2k^2)^2\,\Pi(\lambda_1,\ -\varkappa_1,\ q_1)-\frac{4k^2\beta_0^2(1-\beta_0^2)}{\gamma_0^2}\,\Pi(\lambda_k,\ -\varkappa_k,\ q_k)\Big\}$$

Here, F, E, and Π are elliptic integrals

$$\lambda_j=\arcsin\left(\frac{j^2-\xi^2}{j^2-\beta_0^2}\right)^{1/2},\quad q_j=\left[1-\left(\frac{\beta_0}{j}\right)^2\right]^{1/2},\quad \varkappa_j=\frac{j^2-\beta_0^2}{j^2-\gamma_0^2}\quad (j=1,\ k)$$

The expression for $\sigma_{zz}(\xi)$ with $k < \xi < 1$ is obtained from (5) by eliminating the imaginary terms and the terms with $F(\lambda_k,\ q_k)$, $E(\lambda_k,\ q_k)$, $\Pi(\lambda_k,\ -\varkappa_k,\ q_k)$. Equation (4) for the constant Q can be rewritten in the form:

$$Q=\frac{1}{\Psi(\beta_0^2)}\left[2(p-q_0)\beta_0+2q_0\gamma_0+A\pi-A\int\limits_{\beta_0^2}^{1}\frac{\sqrt{s}-\beta_0}{\sqrt{s}}\,\frac{f(s)[s-(k_s^2-\gamma_0^2)]}{s(s-\gamma_0^2)}\,ds\right]$$
$$\Psi(\beta_0^2)=\pi-I(\beta_0^2)\qquad I(\beta_0^2)=\int\limits_{\beta_0^2}^{1}\frac{\sqrt{s}-\beta_0}{\sqrt{s}}\,\frac{f(s)}{s}\,ds \tag{6}$$

The denominator of the fraction was found by Broberg; the integral in the numerator is similarly evaluated. The expression for Q, found from (6), is substituted in σ_{zz}. As seen from (5), the solution obtained has a singularity of the order of $s^{-1/2}$ (where $s \ll 1$ is the distance from the outside to the tip of the crack). Letting ξ tend to β_0 and setting $m = V_0/c_2$, we obtain an expression for σ_{zz} in the neighborhood of the tip of the moving crack:

$$\sigma_{zz}=F(m,\ \nu)\sqrt{\frac{ct}{s}}\,\{(p-q_0)+q_0\varphi(m,\ \alpha,\ \nu)\} \tag{7}$$
$$F(m,\ \nu)=\frac{\sqrt{1-k^2m^2}\,[4\sqrt{(1-k^2m^2)(1-m^2)}-(m^2-2)^2]}{\sqrt{2}\,m^{3/2}f(m,\ \nu)}$$
$$f(m,\ \nu)=[m^2(1-4k^2)+4k^2]\,K(\sqrt{1-m^2k^2})-4(1-m^2k^2)\,K(\sqrt{1-m^2})$$
$$-\frac{1}{m^2}[m^4-4(1+k^2)m^2+8]\,E(\sqrt{1-m^2k^2})+\frac{8}{m^2}(1-m^2k^2)\,E(\sqrt{1-m^2})$$
$$(m,\ \alpha,\ \nu)=\frac{2\alpha m\sqrt{(1-\alpha^2m^2k^2)(1-\alpha^2)}}{\pi[(\alpha^2m^2-2)^2-4\sqrt{(1-\alpha^2m^2k^2)(1-\alpha^2m^2)}]}\{\alpha^2m^2K(\sqrt{1-m^2k^2})$$
$$-\frac{4}{m^2}E(\sqrt{1-m^2k^2})+\frac{4}{m^2}E(\sqrt{1-m^2})+\frac{k^2(\alpha^2m^2-2)^2}{1-\alpha^2m^2k^2}$$
$$\times\,\Pi\left(-\frac{1-m^2k^2}{1-\alpha^2m^2k^2},\ \sqrt{1-m^2k^2}\right)-4\,\Pi\left(-\frac{1-k^2}{1-\alpha^2k^2},\ \sqrt{1-m^2}\right)$$

The stress at the tip of the crack will be finite if the expression in braces in (7) is equal to zero. Hence we obtain the equation relating the effective stress p and the cohesion forces q_0 with the propagation velocity of the crack

$$p\,/\,q_0=1-\varphi(m,\alpha,\nu) \tag{8}$$

Relation (8) is shown graphically in Fig. 2 for $\nu = 0.333$, where curve I corresponds to $\alpha = 0.5$ and curve II = $\alpha = 0.8$. We present the results of an investigation of Eq. (8) in certain limiting cases. As $\alpha \to 0$, which corresponds to the propagation of cohesion forces over the entire length of the crack, $p/q_0 \to 1$. As $\alpha \to 1$ the end region tends to zero, $p/q_0 \to 0$. The values of $m \ll 1$, using the parametric addition formula for integrals of the third kind [3] are representing the elliptic integrals in series form [4], we obtain

$$\frac{p}{q_0}=A-\frac{1}{B}m^2k^2\ln mk+0\,(m^2k^2) \tag{9}$$

Here

$$A=1-\frac{2}{\pi}\arcsin\alpha$$
$$B=\frac{2\pi k^2(1-k^2)}{\alpha\sqrt{1-\alpha^2}\,[(3-k^4)-4k^2(1-k^2)]}$$

This equation gives the relation between the parameters p, q_0, and m for slow cracks at values of $p/q_0 \gtrsim A$. The solution $P/q_0 = A$ corresponds to the limiting equilibrium of the crack for the static problem.

To clarify the effect of the dynamics of the end region on the velocity of the crack we compare relation (8) with the equation

$$\frac{p}{q_0} = \frac{2\sqrt{(1-\alpha)m}}{\pi F(m, \nu)} \tag{10}$$

which was obtained on the assumption that the stress field created by the cohesion forces near the end of the crack is quasi stationary in character [2]. Curves 1, 2, and 3, in Fig. 2 correspond to Eq. (10) with α equal to 0.5, 0.8, 0.99, respectively.

From a comparison of curves I and 1, II and 2, obtained at the same values of the parameter α, it is clear that as α tends to unity the difference between the dynamic and quasi-stationary solutions decreases, which is attributable to the reduced rate of growth of the end region. In [2] it was found that for a given material there is a certain minimum rate of uniform crack propagation. This was because of the assumption that the end region d expands at a rate v that does not depend on the propagation velocity of the crack itself:

$$d = vt \tag{11}$$

The relation corresponding to (11) and quasi stationarity on the stress field due to the cohesion forces

$$\frac{p}{q_0} = \frac{2\sqrt{v/c_2}}{\pi F(m, \nu)}$$

is represented by the dashed line in Fig. 2 for $v/c_2 = 10^{-2}$. Here, it is assumed that $d = (1-\alpha)Vt$. Then the crack propagation velocity may be arbitrary, bounded above by c_s as $\alpha \to 1$. Given assumption (11), which corresponds to $\alpha = 1 - v/V$, the solution obtained also leads to the existence of a minimum crack propagation velocity for a given material.

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